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A PROBABILISTIC APPROACH TO A BOUNDARY LAYER PROBLEM

BY

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ABSTRACT. An elliptic second order linear operator is approximated by the transition operator of a Markov chain, and the solution to the corresponding approximate boundary value problem is expanded in terms of a small parameter, up to the first order term. In characterizing the boundary values of the first order term in the expansion, a problem of a boundary layer arises, which is treated by probabilistic methods.

1. Introduction. Consider the boundary value problem:

$$(1.1)\qquad \begin{aligned} Lu(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) \\ + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} u(x) = -g(x) \\ \text{for } x \in G = \{x: \phi(x) < 0\}, x \in R^n, \end{aligned}$$

and

$$u(x) = b(x) \quad \text{for } x \in \partial G = \{x: \phi(x) = 0\},$$

where $\phi(x) = 0$ is the boundary of a closed, bounded, connected domain G such that $\nabla \phi(x) \neq 0$ for all $x \in \partial G$. Assume that $a_{ij}(x)$, $b_j(x)$, $b(x)$, $g(x)$, $\phi(x)$ are in C_b^∞ and that the matrix $(a_{ij}(x))^{n \times m}$ is uniformly positive definite. Let $p_h(x, y)$ be a transition density of a Markov chain for which the jumps are uniformly bounded by $ch^{1/2}$, where c is a fixed positive constant and h a small parameter. We define a transition operator Π_h on C_b to be

$$(\Pi_h f)(x) = \int p_h(x, x') f(x')\, dx'.$$

Assume that

$$(1.2)\qquad \lim_{h \to 0} \frac{(\Pi_h f)(x) - f(x)}{h} = Lf(x)$$

uniformly on compacts of R^n for each $f \in C_b^2$. We formulate the approximate problem of (1.1) to be

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$$(1.3)\qquad \begin{aligned} (\Pi_h u_h)(x) - u_h(x) &= -hg(x), \qquad x \in \bar{G}, \\ u_h(x) &= F(x), \qquad x \in \bar{G}', \end{aligned}$$

where $F(x)$ is a continuous extension of the boundary values $b(x)$ into $\bar{G}' = \{x: \phi(x) > 0\}$. From the probabilistic point of view the solution to (1.3) is

$$u_h(x) = E_x^h F(x_{\tau_G}^h) + hE_x^h\left(\sum_{j=0}^{\tau_G - 1} g(x_j^h)\right)$$

where τ_G is the first exit time from $\bar{G}$ of a Markov chain for which (1.2) holds. We obtain an asymptotic expansion of the form

$$(1.4)\qquad u_h(x) = u(x) + \sqrt{h}\, u_1(x) + o(\sqrt{h}\,)$$

where $u(x)$ is the solution to (1.1) and $u_1(x)$ is the solution to another boundary value problem, namely:

$$Lu_1(x) = -L_1 u(x), \qquad x \in G;$$

where L_1 is a differential operator determined by the Markov chain, and $u_1(x) = b_1(x)$ for $x \in \partial G$, where b_1 is to be determined. To see the kind of difficulties one can anticipate in characterizing $b_1(x)$, suppose we postulate that

$$(\Pi_h f)(x) = (I + hL + h^{3/2}L_1)(f)(x) + o(h^{3/2}),$$

$f \in C_b^\infty$; then the approximate equation (1.3) can be written in the form

$$(1.5)\qquad (L + h^{1/2}L_1)u_h(x) = -g(x) + o(h^{1/2}).$$

Thus (1.3) can be viewed as an equation involving a singular perturbation of the operator L.

As an example of how the operator L_1 is determined by the Markov chain, consider a probability distribution $F(y)$ with zero mean, variance 1, third moment $m_3 \neq 0$ and density $f(x)$; then for $a(x) \geqslant \delta > 0$, $b(x)$ in C^∞

$$\frac{1}{\sqrt{ha(x)}} F\left[\frac{y - x - hb(x)}{\sqrt{ha(x)}}\right]$$

is a distribution with mean $x + hb(x)$ and variance 1. We make a change in variable, $y = \sqrt{ha(x)}\; z + x + hb(x)$ and define Π_h as

$$\Pi_h u_h(x, y) = \int u_h\left(x, x + b(x)h + \sqrt{a(x)h}\; z\right) f(z)\, dz.$$

Now expanding formally by a Taylor expansion $u_h(x, y)$ with respect to y at $y = x$ and collecting terms in powers of $\sqrt{h}$, we obtain:

$$\Pi_h u_h(x,y)$$

$$= \left[I + h\left(\frac{a(x)}{2} \frac{\partial^2}{\partial y^2} + b(x) \frac{\partial}{\partial y} \right) + h^{3/2}\left(\frac{m_3}{3!} (a(x))^{3/2} \frac{\partial^3}{\partial y^3} \right) \right] u_h(x,y)$$

$$+ o(h^{3/2}).$$

Thus

$$L_1 = \frac{m_3}{3!} (a(x))^{3/2} \frac{\partial^3}{\partial y^3}.$$

We know from singular perturbation theory that, in characterizing $b_1(x)$, we can anticipate a boundary layer problem. Employing probability methods, we treat the boundary layer problem and obtain the characterization of $b_1(x)$.

2. **The asymptotic expansion.** Since no complications arise in generalizing our results from dimension $n = 2$ to $n > 2$, we shall limit our discussion to the two dimensional case. It is known that the $\lim_{h\to 0} u_h(x) = u(x)$ for $x \in G$, see [4]. We begin the characterization of the first order term by proving the following:

LEMMA 1. *If*

$$\lim_{h\to 0} \frac{u_h(x) - u(x)}{h^{1/2}} = u_1(x)$$

exists uniformly on compacts in G, then it is of the form $u_1(x) = \bar{u}_1(x) + \hat{u}_1(x)$, *where* $\hat{u}_1(x)$ *is the solution to* $L\hat{u}_1(x) = -L_1 u(x)$ *with zero boundary conditions, and* $\bar{u}_1(x)$ *is harmonic for L inside G.*

PROOF. Let $U(x)$ be a smooth extension of $u(x)$ into $\overline{G}'$, i.e., let the inward derivatives of $u(x)$ agree with the corresponding outward derivatives of $U(x)$ on ∂G. We note that $u(x)$ possesses these inward derivatives since it is known that if the data are smooth then so is the solution in the case of elliptic boundary value problem [1]. It is easy to check that

$$U(x_n^h) - \sum_{j=0}^{\tau_G - 1} (\Pi_h - I) U(x_j^h)$$

is a martingale, where $x, x_1^h, x_2^h, x_n^h, \ldots$ is a Markov chain satisfying the assumptions of §1. Substituting

$$(\Pi_h - I)U(x) = -hg(x) - h^{3/2} g_1(x) + o(h^{3/2}),$$

where $L_1 u(x) = g_1(x)$, in the martingale and taking into account that the expectation of a martingale is the same for all parameter values, that the expectation of the martingale at $t = 0$ is $U(x)$ and Doob's stopping theorem, we have

$$E_x^h\left(U\left(x_{\tau_G}^h\right) - \sum_{j=0}^{\tau_G-1} (\Pi_h - I)U\left(x_{\tau_G}^h\right)\right)$$

$$= E_x^h U\left(x_{\tau_G}^h\right) + hE_x^h \sum_{j=0}^{\tau_G-1} g\left(x_n^h\right)$$

$$+ h^{3/2}E_x^h \sum_{j=0}^{\tau_G-1} g_1\left(x_j^h\right) + E_x^h(\tau_G)o(h^{3/2})$$

$$= U(x).$$

Since $E_x^h(\tau_G) < \text{const}/h$ for a discrete time Markov chain, see [2],

$$\frac{u_h(x) - U(x)}{\sqrt{h}} = \frac{1}{\sqrt{h}}\left(E_x^h\left(F\left(x_{\tau_G}^h\right) - U\left(x_{\tau_G}^h\right)\right)\right) - hE_x^h \sum_{j=0}^{\tau_G-1} g_1\left(x_j^h\right) + o(1).$$

But

$$\lim_{h\to 0} hE_x^h\left(\sum_{j=0}^{\tau_G-1} g_1\left(x_j^h\right)\right) = E_x\int_0^{\tau_G} g_1(x_t)\,dt,$$

where x_t is a diffusion process generated by the operator L. This limit (let us denote it by $\hat{u}_1(x)$) is of course the solution to the Poisson equation $L\hat{u}_1(x) = -g_1(x)$ with zero boundary conditions. Now let

$$\bar{u}_1^h(x) = \frac{1}{h^{1/2}} E_x^h\left(F\left(x_{\tau_G}^h\right) - U\left(x_{\tau_G}^h\right)\right).$$

By assumption $\lim_{h\to 0}\bar{u}_1^h(x) = \bar{u}_1(x)$ exists for $x \in G$. Let D be a compact subset of G. Then by the strong Markov property

$$\bar{u}_1^h(x) = E_x^h\bar{u}_1^h\left(x_{\tau_D}^h\right) = \int \bar{u}_1^h(x)H_h^D(x, dy),$$

where τ_D is the first exit time from D and $H_h^D(x, dy)$ is the exit distribution from D. Since the $\lim_{h\to 0} H_h^D(x, dy)$ is the harmonic measure on ∂D, it follows that $\bar{u}_1(x)$ is harmonic in D. This completes the proof of Lemma 1.

It remains to prove that the hypothesis of Lemma 1 is satisfied and to obtain the boundary values of $\bar{u}_1(x)$. We begin by proving

LEMMA 2.

$$\lim_{\substack{h\to 0\\ x\to b\in\partial G}} \left|\bar{u}_1^h(x) - W(b)\frac{1}{\sqrt{h}} E_x^h\left(\phi\left(x_{\tau_G}^h\right)\right)\right| = 0$$

where

$$W(x) = \frac{\langle \nabla(F(x) - U(x)), \nabla\phi(x)\rangle}{\|\nabla\phi(x)\|^2}$$

($\langle\cdot,\cdot\rangle$ *denotes inner-product*).

PROOF. Let $V(x) = F(x) - U(x)$; then

$$\bar{u}_1^h(x) = (1/h^{1/2}) E_x^h V\left(x_{\tau_G}^h\right);$$

if we expand $V(x)$ in the normal direction to $\phi(x) = 0$, we obtain

$$V(x) = \frac{\langle V(x), \nabla \phi(x) \rangle}{\| \nabla \phi(x) \|^2} \phi(x) + O\left(\phi^2(x)\right).$$

Since the jumps of the Markov chain are uniformly bounded by $ch^{1/2}$, we have $O(\phi^2(x_{\tau_G}^h)) = O(h)$. Thus

$$\bar{u}_1^h(x) = \frac{1}{\sqrt{h}} E_x^h\left(W\left(x_{\tau_G}^h\right)\phi\left(x_{\tau_G}^h\right)\right) + O\left(h^{1/2}\right).$$

Now for b on ∂G

$$\lim_{\substack{h \to 0 \\ x \to b}} \frac{1}{\sqrt{h}} \left| E_x^h\left(W\left(x_{\tau_G}^h\right)\phi\left(x_{\tau_G}^h\right)\right) - W(b) E_x^h \phi\left(x_{\tau_G}^h\right) \right|$$

$$= \lim_{\substack{h \to 0 \\ x \to b}} \frac{1}{\sqrt{h}} \left| E_x^h\left[\phi\left(x_{\tau_G}^h\right)\left(W\left(x_{\tau_G}^h\right) - W(b)\right)\right] \right|$$

$$\leqslant \lim_{\substack{h \to 0 \\ x \to b}} k\, E_x^h \left| W\left(x_{\tau_G}^h\right) - W(b) \right| = 0.$$

The last inequality follows from the assumption that the jumps of the Markov chain are uniformly bounded by $ch^{1/2}$. Assuming the continuity of $W(x)$, the last equality follows from the fact that $V_n(x) = E_x^h[|W(x_{\tau_G}^h) - W(b)|]$ converges uniformly in $x \in \bar{G}$ to the solution of $LV = 0$ with $V = |W(\cdot) - W(b)|$ on ∂G. This completes the proof of Lemma 2.

Thus, by Lemma 2, to identify the boundary value of $\bar{u}_1(x)$ at b, we must identify

$$\lim_{x \to b} \lim_{h \to 0} \frac{1}{h^{1/2}} E_x^h \phi\left(x_{\tau_G}^h\right).$$

To this end we require several preliminary lemmas.

LEMMA 3. *Assume the transition density $p_h(x, y)$ is such that*

$$\lim_{\substack{h \to 0 \\ x \to b}} h p_h\left(x, x + h^{1/2} z\right) = p_0^b(z),$$

where $p_0^b(z)$ is a bivariate probability density. Then for f in C_b^∞

$$\lim_{\substack{h \to 0 \\ x \to b}} \iint_{R^2} \frac{f(\phi(y) - \phi(x))}{\sqrt{h}} p_h(x, y)\, dy = \int_R f(s) p^b(s)\, ds,$$

where $p^b(s)$ is the marginal density of $\langle \nabla \phi(b), Z \rangle$ and Z is a random vector with density $p_0^b(z)$.

PROOF. By a Taylor expansion

$$\phi(y) = \phi(x) + \langle \nabla \phi(x), (y - x)\rangle + O(\|y - x\|^2).$$

Thus

$$\iint f\left(\frac{\phi(y) - \phi(x)}{\sqrt{h}}\right) p_h(x, y)\, dy$$

$$= \iint f\left(\frac{\langle \nabla \phi(x), (y - x)\rangle + O(\|y - x\|^2)}{\sqrt{h}}\right) p_h(x, y)\, dy$$

$$= \iint \left(f\left(\frac{1}{\sqrt{h}} \langle \nabla \phi(x), (y - x)\rangle\right) + \frac{1}{\sqrt{h}} O(\|y - x\|^2)\right) p_h(x, y)\, dy$$

where $O(\|y - x\|^2) \leq k\|y - x\|^2$. But assumption (1.2) implies

$$\lim_{h\to 0} \frac{1}{h} \iint \|y - x\|^2 p_h(x, y)\, dy = \operatorname{trace}(a_{ij}(x));$$

thus

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \iint \|y - x\|^2 p_h(x, y)\, dy = 0.$$

Taking $z = (y - x)/\sqrt{h}$ we have

$$\iint_{R^2} f\left(\frac{1}{\sqrt{h}} \langle \nabla \phi(x), (y - x)\rangle\right) p_h(x, y)\, dy$$

$$= \iint_{R^2} f(\langle \nabla \phi(x), z\rangle) h p_h(x, x + \sqrt{hz}\,)\, dz.$$

But

$$\lim_{\substack{h\to 0\\ x\to b}} h p_h(x, x + \sqrt{h}\, z) = p_0^b(z).$$

Therefore,

$$\lim_{\substack{h\to 0\\ x\to b}} \iint_{R^2} f(\langle \nabla \phi(x), z\rangle) h p_h(x, x + \sqrt{h}\, z)\, dz = \int_R f(s) p^b(s)\, ds.$$

This completes the proof of Lemma 3.

LEMMA 4. *Let*

$$z_h(n) = (1/\sqrt{h}\,)\phi(x_n^h),$$

where $\{x_n^h\}$ *is a Markov chain starting at* $x \in G$ *with transition density* $p_h(x, y)$. *Then for every* k, $z_h(1) - z_h(0), z_h(2) - z_h(1), \ldots, z_h(k) - z_h(k - 1)$ *converge weakly to* $z(1) - z(0), z(2) - z(1), \ldots, z(k) - z(k - 1)$ *as* $h \to 0$ *and* $x \to b$ *where* $z(n)$ $(n = 0, 1, 2, \ldots)$ *is a random walk with transition density* $p^b(s)$.

PROOF. To show that any finite number of increments of $z_h(n)$ ($n = 0, 1, 2, \dots$) converge weakly to the corresponding finite number of increments of $z(n)$ ($n = 0, 1, 2, \dots$), it suffices to show that for any k,

$$\lim_{\substack{h\to 0\\ x\to b}} E_x^h\big(f_1(z_h(1) - z_h(0))f_2(z_h(2) - z_h(1))\cdots f_k(z_h(k) - z_h(k-1))\big)$$
$$= E_b\big(f_1(z(1) - z(0))f_2(z(2) - z(1))\cdots f_k(z(k) - z(k-1))\big)$$

where f_j ($j = 1, 2, \dots k$) are any continuous, bounded functions. Now

$$E_x^h f_1(z_h(1) - z_h(0))f_2(z_h(2) - z_h(1))\cdots f_k(z_h(k) - z_h(k-1))$$
$$= \int_{R^{2k}} f_1\left(\frac{\phi(x_1) - \phi(x)}{\sqrt{h}}\right) f_2\left(\frac{\phi(x_2) - \phi(x_1)}{\sqrt{h}}\right)\cdots f_k\left(\frac{\phi(x_k) - \phi(x_{k-1})}{\sqrt{h}}\right)$$
$$\times p_h(x, x_1)p_h(x_1, x_2)\cdots p_h(x_{k-1}, x_k)\, dx_1\, dx_2 \cdots dx_k.$$

We shall evaluate one factor at a time; in the limit this will permit us to write the above expectation as a product of expectations. Evaluating the limit of the kth factor first, i.e., by Lemma 3

$$\lim_{\substack{h\to 0\\ x_{k-1}\to b}} \iint_{R^2} f_k\left(\frac{\phi(x_k) - \phi(x_{k-1})}{\sqrt{h}}\right) dx_k = \int_R f(s)p^b(s)\, ds, \qquad x_k \in R^2.$$

Similarly, by Lemma 3 we evaluate the limit of the $(k-1)$th factor, and again obtain $\int_R f(s)p^b(s)\, ds$. Continuing to evaluate each consecutive factor in the same manner, we obtain

$$\lim_{\substack{h\to 0\\ x_i\to b\\ i=k-1,k-2,\dots,0}} E_x^h f_1(z_h(1) - z_h(0))\cdots f_k(z_h(k) - z_h(k-1))$$
$$= \int_R f_1(s)p^b(s)\, ds \cdots \int_R f_k(s)p^b(s)\, ds.$$

This implies that $z(i) - z(i-1)$ ($i = 1, 2, \dots, k$) are independent and identically distributed, and thus $z(n)$, $n = 0, 1, 2, \dots,$ is a random walk. This completes the proof of Lemma 4.

LEMMA 5.

$$\lim_{\substack{h\to 0\\ x\to b\\ \phi(x)/\sqrt{h}\to -s}} \frac{1}{\sqrt{h}} E_x^h\left(\phi\left(x_{\tau_G}^h\right)\right) = E_{-s}^b\left(z(\tau_G)\right)$$

where x_n^h is a Markov chain with transition density $p_h(x,y)$ and $z(n)$ is a random walk with transition density $p^b(s)$.

REMARK. The

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} E_x^h\left(\phi\left(x_{\tau_G}^h\right)\right)$$

is nonuniform near the boundary of G; thus great care must be taken in how we let $h \to 0$ and $x \to b$. Above we let $h \to 0$ and $x \to b$ in such a way that

$$\phi(x)/\sqrt{h} \to -s$$

then in order to evaluate

$$\lim_{x\to b} \lim_{h\to 0} \frac{1}{\sqrt{h}} E_x^h\left(\phi\left(x_{\tau_G}^h\right)\right)$$

we shall let $s \to \infty$. $E_{-s}^b(z(\tau_G))$ is the expected overshoot in the normal direction over the boundary at $x = b$ of the random walk $z(n)$ starting at $-s$.

PROOF. We note that

$$z_h(\tau_G) = (1/\sqrt{h})\phi\left(x_{\tau_G}^h\right);$$

thus

$$z_h(0) = \phi(x)/\sqrt{h},$$

and thus

$$(1/\sqrt{h})E_x^h\left(\phi\left(x_{\tau_G}^h\right)\right) = E_{\phi(x)/\sqrt{h}}\left(z_h(\tau_G)\right).$$

By Lemma 4 the finite dimensional distributions of the stochastic process $z_h(n)$ $(h = 0, 1, 2, \dots)$ converge weakly to the finite dimensional distributions of the random walk $z(n)$ $(n = 0, 1, 2, \dots)$. Since $\tau_G = \inf_n\{n: z_h(n) > 0\}$ is finite almost surely, it follows by a standard weak convergence argument that

$$\lim_{\substack{h\to 0\\ x\to b\\ \phi(x)/\sqrt{h}\to -s}} E_{\phi(x)/\sqrt{h}}^h\, z_h(\tau_G) = E_{-s}^b\left(z(\tau_G)\right).$$

This completes the proof of Lemma 5.

REMARK. Assuming that $p_0^b(z)$ in Lemma 3 is such that $\iint z p_0^b(z)\, dz = 0$, then the random walk $z(n)$ will be persistent and by the Renewal Theorem $\lim_{s\to\infty} E_{-s}^b\, z(\tau_G)$ exists; it is shown in [3] that for a persistent, nonlattice random walk the limiting exit distribution exists.

We shall assume:

$$\text{(2.1)} \qquad \lim_{s\to\infty} E_{-s}^b\, z(\tau_G) \text{ exists uniformly in } b \in \partial G.$$

In (2.1) we are assuming that if two boundary points are close then so are the expected overshoots, in the normal directions to the boundary at these points, of the random walk $z(n)$. For this it is enough to assume sufficient smoothness of the boundary and the transition density.

Denote $E_{-s}^b(z(\tau_G))$ by $V_s(b)$ and $\lim_{s\to\infty} V_s(b)$ by $V(b)$. Our next objective is to show that $\lim_{h\to 0} \bar{u}_1^h(x) = \bar{u}_1(x)$ exists, $L\bar{u}_1 = 0$, and $\bar{u}_1(x)$ takes on the boundary value $W(b)V(b)$ at $b \in \partial G$. To this end we first prove the following

LEMMA 6. *Let P_n be a sequence of probability measures on a measurable space $(X, \mathfrak{F})$ converging weakly to a probability measure P, where X is a complete separable metric space and $\mathfrak{F}$ its Borel field. Let C be a closed set in $\mathfrak{F}$ such that $P(C) = 1$; let $f_n(x)$ be a sequence of measurable functions on X and $f(x)$ a measurable function on X. Furthermore assume*

(1) $|f_n(x)| \leqslant K$ *for all n, and*

(2) $\lim_{n\to\infty; x_n\to x\in C} |f_n(x_n) - f(x)| \leqslant \delta$;

then

$$\overline{\lim_{n\to\infty}} \left|\int_X f_n(x)\, dP_n(x) - \int_X f(x)\, dP(x)\right| \leqslant \delta.$$

PROOF. Suppose the conclusion is not true; then there exist a subsequence $f_{n_j}(x)$ and P_{n_j} such that

$$\overline{\lim_{n_j}} \left|\int_X f_{n_j}(x)\, dP_{n_j} - \int_X f(x)\, dP(x)\right| = \delta' > \delta.$$

But P_{n_j} converges weakly to P; thus by a theorem of Skorohod there exist a probability space (Ω, Σ, Π) and random variables $x_{n_j}(\omega)$, $x(\omega)$ such that $\Pi(x_{n_j}^{-1}(A)) = P_{n_j}(A)$, and $\Pi(x^{-1}(A)) = P(A)$ for all $A \in \mathfrak{F}$; further, $x_{n_j}(\omega)$ converges in probability to $x(\omega)$, and there exists a further subsequence $x_{n_{j_k}}(\omega)$ which converges to $x(\omega)$, almost surely. Thus by assumption (2)

$$\overline{\lim_{n\to\infty}} \left|f_{n_{j_k}}\left(x_{n_{j_k}}(\omega)\right) - f(x(\omega))\right| \leqslant \delta$$

a.e., and by the Bounded Convergence Theorem

$$\overline{\lim_{n_{j_k}}} \left|\int_\Omega f_{n_{j_k}}\left(x_{n_{j_k}}(\omega)\right) d\Pi(\omega) - \int_\Omega f(x(\omega))\, d\Pi(\omega)\right| \leqslant \delta.$$

But this contradicts the assumption that

$$\overline{\lim_{n_j}} \left|\int_X f_{n_j}(x)\, dP_{n_j}(x) - \int_X f(x)\, dP(x)\right| = \delta' > \delta.$$

This completes the proof of Lemma 6.

LEMMA 7. $\lim_{h\to 0}\bar{u}_1^h(x) = \bar{u}_1(x)$ *exists*; $L\bar{u}_1(x) = 0$ *for* $x \in G$ *and* $\bar{u}_1(b) = W(b)V(b)$, $b \in \partial G$.

PROOF. Let $\tau_{s,h}$ be the first exit time of a Markov chain $\{x_n^h\}$ with transition density $p_h(x, y)$ from $\{x: \phi(x) < -sh^{1/2}\}$, $s > k$, where $kh^{1/2}$ is the uniform bound on the jumps of $\phi(x_n^h)$. Then by assumption (2.1), Lemma 2 and Lemma 5 we have

$$\lim_{\substack{h\to 0\\ x\to b\\ \phi(x)/\sqrt{h}\to s'}} \left|\bar{u}_1^h(x) - W(b)V_{s'}(b)\right| = 0,$$

where $V_{s'}(b) = E^b_{-s'}(z(\tau_G))$. From this it follows that for every $b \in \partial G$,

$$\begin{aligned}
&\lim_{\substack{h\to 0\\ x\to b\\ -s\sqrt{h} < \phi(x) < -s\sqrt{h}+c\sqrt{h}}} \left|\bar{u}_1^h(x) - W(b)V(b)\right| \\
&\quad < \sup_{s-c<s'<s} \left|W(b)V_{s'}(b) - W(b)V(b)\right| \\
&\quad \leqslant \sup_b \left|W(b)\right| \sup_{s-c<s'<s} \sup_b \left|V_{s'}(b) - V(b)\right| \\
&\quad = \sup_b \left|W(b)\right| \mathcal{E}(s-c)
\end{aligned}$$

where $\mathcal{E}(s) = \sup_{s'>s} \sup_b |V_{s'}(b) - V(b)|$. By our assumption (2.1) $\mathcal{E}(s) \to 0$ as $s \to \infty$. Now by the strong Markov property

$$\bar{u}_1^h(x) = E_x^h \bar{u}_1\left(x^h_{\tau_{s,h}}\right) = \iint_D \bar{u}_1^h(x) H_h^s(x, dy),$$

$$D = \left\{x\colon -sh^{1/2} \leqslant \phi(x) \leqslant -sh^{1/2} + ch^{1/2}\right\},$$

where $H_h^s(x, dy)$ is the exit distribution from $\{s\colon -sh^{1/2} \geqslant \phi(x)\}$. It is well known that for regular boundary points $b \in \partial G$,

$$\lim_{\substack{h\to 0\\ x\to b}} H_h^s(x, dy) = H(x, db),$$

where $H(x, db)$ is the harmonic measure on ∂G. Setting $C = \partial G$, $f_n = \bar{u}_1^h$, $f = \bar{u}$, $P_n = H_h^s$, $P = H$ and $\mathcal{E}(s) = \delta$ in Lemma 6, and noting that all the assumptions of Lemma 6 are met, we can conclude by Lemma 6 that

$$\begin{aligned}
&\overline{\lim_{h\to 0}} \left|\bar{u}_1^h(x) - \bar{u}_1(x)\right| \\
&\quad = \overline{\lim_{h\to 0}} \left| \iint_D \bar{u}_1^h(x) H_h^s(x, dy) - \int_{\partial G} W(b)V(b)H(x, db)\right| \\
&\quad < \mathcal{E}(s-c).
\end{aligned}$$

But $\bar{u}_1^h(x)$ and $\bar{u}_1(x)$ are independent of s, and $\mathcal{E}(s-c) \to 0$ as $s \to \infty$. This completes the proof of Lemma 7.

Recalling that $u_1(x) = \hat{u}_1(x) + \bar{u}_1(x)$ where $\hat{u}_1(x)$ satisfies the Poisson equation, $L\hat{u}_1 = -g_1$, with zero boundary condition and $\bar{u}_1(x)$ is harmonic for L inside G with the boundary values given by Lemma 7. We can now summarize our results by the following

THEOREM. *Let*

$$A = \{x: \phi(x) \leqslant d < 0\},$$

$$V(b) = \lim_{\substack{h \to 0 \\ x \to b \\ \phi(x)/\sqrt{h} \to \infty}} \frac{1}{\sqrt{h}} E_x^h \phi\left(x_{\tau_G}^h\right)$$

and

$$W(x) = \frac{\langle \nabla (F(x) - U(x)), \nabla \phi(x) \rangle}{\| \nabla \phi(x) \|^2},$$

then

$$\lim_{h \to 0} \sup_{x \in A} \left| \frac{u_h(x) - u(x)}{\sqrt{h}} - u_1(x) \right| = 0,$$

where $u_1(x)$ *is the solution to* $Lu_1(x) = -g_1(x)$ *with boundary values* $u_1(b) = W(b)V(b), b \in \partial G$.

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